ISRAEL JOURNAL OF MATHEMATICS, Vol. 60, No. 1, 1987

STRICTLY ERGODIC MODELS AND TOPOLOGICAL MIXING FOR Z^2-ACTION

BY

A. ROSENTHAL[†]

Laboratoire de Probabilites, Universite de Paris VI, Tour 56,
4, Place Jussieu, 75230 Paris, Cedex 05, France; and
Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem, Israel

ABSTRACT

We prove that any free ergodic Z^2-action has a strictly ergodic and topologically mixing model.

I. Introduction

The action of a group G by homeomorphisms of a compact metric space X is said to be strictly ergodic, if there is a unique Borel probability measure ν, fixed by the action and $\nu(U) > 0$ for every non-empty open set $U \subset X$. A beautiful result due to R. Jewett [J] and W. Krieger [K] says that for $G = Z$, any ergodic action is isomorphic to a strictly ergodic system. Recently B. Weiss [W] extended this result to any ergodic free action of an elementary amenable group. Using Weiss's methods, and under his guidance we were able to obtain the result for any ergodic free action of a general discrete amenable group [R1]. In the case of a Z-action one of Weiss's students, E. Lehrer [L], was able to prove that any ergodic Z-action is isomorphic to a strictly ergodic and topologically mixing system (see definition below). The purpose of our work is to prove that Lehrer's result extends also to any free, ergodic Z^2-action. Lehrer's work was done before Weiss's extension and new proof of the Jewett–Krieger theorem. His method does not generalize to Z^2. On the other hand our method can be easily adapted to any discrete amenable group.

[†] *Present address*: 17A, Ha-Lamed Hey St., 93661 Jerusalem, Israel.

Received December 15, 1985 and in revised form May 27, 1987

II. Statement of the results

DEFINITION. Suppose there is a Z^2-action by homeomorphisms on a compact metric space X, the action is said to be topologically mixing if for any non-empty open sets U, V in X, there exists a finite set $K \subset Z^2$ such that for any g not in K, $(gU \cap V) \neq \emptyset$.

Our goal in the following work is to prove:

THEOREM 1. *Suppose we are given a free ergodic measure preserving Z^2-action on a Lebesgue space $(Y, \mathcal{B}, \lambda)$. There exists a strictly ergodic Z^2-action of a compact metric space X with unique invariant measure v that is topologically mixing and measure isomorphic to $(Y, \mathcal{B}, \lambda)$.*

In the sequel we are given a free ergodic measure preserving Z^2-action on $(Y, \mathcal{B}, \lambda)$. We will use the following lemma, an exercise using Rohlin's lemma (see E. Lehrer [L] in the Z-case):

LEMMA 2. *For every sequence (finite or infinite) $H_1, H_2, \ldots, H_m, \ldots$ of finite subsets of Z^2 and $\alpha > 0$ there are subsets $A_1, A_2, \ldots, A_m, \ldots$ of Y such that:*

(i) $\lambda(A_m) > 0$, *for any* m,

(ii) $\lambda(\bigcup_{m=1} \bigcup_{g \text{ in } H_m} gA_m) < \alpha$,

(iii) *the sets* gA_m (g *in* H_m, $1 \leqq m$) *are pairwise disjoint.*

DEFINITION. Let $S_n = \{(i,j) \text{ in } Z^2;\ |i| \leqq n \text{ and } |j| \leqq n\}$.

DEFINITION. A partition P of Y is said to be topologically mixing for the Z^2-action if for any n, and every two atoms q, r of $\bigvee_{c \text{ in } S_n} cP$, we have $\lambda(gq \cap r) > 0$, if g is not in a finite subset $K(q, r)$ of Z^2.

DEFINITION. A partition P of Y is said to be uniform if for every n and every α if $\bigvee_{c \text{ in } S_n} cP = (p_1, p_2, \ldots, p_{s_n})$ there exists M such that for almost every y in Y, we have:

$$\sum_{i=1}^{i=s_n} |1_{p_i}(cy) - \lambda(p_i)| \leqq \alpha. \tag{1}$$

In the case where (1) is true for a given n, M, α we will say that P is (n, M, α) uniform.

DEFINITION. If $P = (p_1, p_2, \ldots, p_l)$ is a partition and D is a subset of Z^2, the D-P-name of some point y is the element of $\{1, 2, \ldots, I\}^D : (\beta_d(y))_{d \text{ in } D}$, such that for any d in $D : dy$ is in $p_{\beta_d(y)}$.

The main work will consist of proving:

THEOREM 3. *For every finite partition P and every δ, there exists a partition $\bar{P}$ uniform and topologically mixing such that $d(P, \bar{P}) \leqq \delta$.*

COROLLARY. *There exists a sequence $Q_1 \subset Q_2 \subset \cdots \subset Q_n \cdots$, such that $\bigvee_{n \text{ in } N} Q_n = \mathscr{B}$ (the entire σ-algebra), and for every n, Q_n is uniform and topologically mixing.*

PROOF. It is similar to the proof of corollary 12 of [R2]. (Instead of, as in the proof of Theorem 3 (see below), building a single partition, we build $Q_1^{(1)}$ at step 1, then $Q_1^{(2)} \subset Q_2^{(2)}$ at step 2 and so on with $Q_n = \lim_j Q_n^{(j)}$.)

PROOF OF THEOREM 1. This comes immediately from the above corollary; it is just a translation of it (see [H–R] or [R2]).

REMARK. In the case of a general discrete, amenable group we will briefly indicate at the end of Section III the slight modifications needed for the proof to work in that case.

III. Construction of a uniform, mixing partition and Proof of Theorem 3

PROOF. Using the theorem proved in [W] or [R1] stating that for every finite partition P and every δ, there exists a uniform partition P' such that $d(P, P') < \delta$, we will suppose that P is a uniform partition. Secondly, as it was proved in [R2] (see corollary 12 there), we can suppose that there exists a sequence $(P_i)_{i \geqq 0}$ of uniform partitions such that $P_0 = P \subset P_1 \subset \cdots \subset P_i$ and $\bigvee_{i \geqq 0} P_i = \mathscr{B}$. The proof is then conducted by an inductive procedure. Let $(\alpha_n)_{n \geqq 1}$ be a fixed sequence of real numbers such that $\Sigma_{n=1}^{n=+\infty} \alpha_n < \delta$. We will start from $P_0 = P$ and construct P_i so that $d(P_i, P_{i+1}) \leqq \alpha_{i+1}$ and thus the limiting partition $\bar{P}$ satisfies $d(P, \bar{P}) < \delta$.

Preliminary to step 1: Because P is uniform, we can find n_1 such that P is $(1, n_1, \alpha_1/10)$ uniform. Let us also choose M_1 with $n_1/M_1 \leqq \alpha_1/10$. We then choose n_2 bigger than M_1 such that P is $(2, n_2, \alpha_2/10)$ uniform and M_2 with $n_2/M_2 \leqq \alpha_2/10$.

DEFINITION. $J = (j_1, j_2)$ in Z^2 is said to be between I' and I'', (I', I'') in N^2

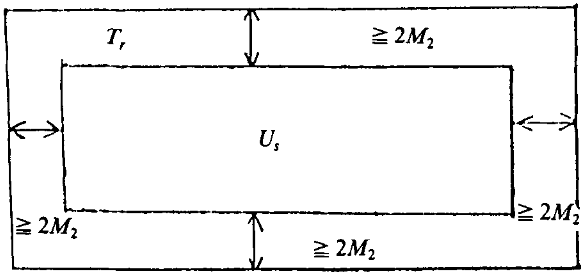

Fig. 1.

if $I' \leqq \mathrm{Max}(|j_1|, |j_2|) \leqq I''$. P is said to be (k, I', I'') mixing if for any atom q, r in $\bigvee_{c \text{ in } S_k} cP$ and any J between I' and I'', $\lambda(Jq \cap r) > 0$.

Step 1: Let $I_1 = 10M_1$, $I_2 = 10M_2$. Applying Lemma 2 with $\alpha = \alpha_1/2$, we find a sequence of Rohlin towers of size $r = (r_1, r_2)$: T_r, for any r between $I_1 + 10M_2$ and $I_2 + 10M_2$, that is by definition:

$$T_r = \bigcup_{\substack{c=(i,j) \\ 0 \leqq i \leqq r_1 - 1 \\ 0 \leqq j \leqq r_2 - 1}} cB_r$$

and all the cB_r are disjoint (for all the different (c, r)). Finally $\lambda(\bigcup_r T_r) \leqq \alpha_1/2$. Let us fix some r, and thus $T_r: r = (r_1, r_2) = (s_1 + 10M_2, s_2 + 10M_2)$, let $s = (s_1, s_2)$ (we only use those r with $\mathrm{Min}(r_1, r_2) \geqq 10M_2$). In T_r it is easy to find a tower U_s of size s (see above definition) whose boundary is at least $2M_2$ apart from that of T_r (see Fig. 1). Furthermore, because $\bigvee_i P_i = \mathscr{B}$, replacing if necessary every B_r by a set in some uniform partition, as close to B_r as wanted, we may suppose that all the sets considered belong to some uniform partition P_{i_1}. Supose now that $P = (p_1, p_2, \ldots, p_t)$ for each I, $1 \leqq I \leqq t = |P|$, $\lambda(p_I) > 0$. Let I, $1 \leqq I \leqq t$ be fixed. Because $\lambda(p_I) > 0$ and because almost every y is $(1, n_1, \alpha_1/10)$ "good" for P (that is, satisfies (1) for n_1 and P) there exists y_1, y_1 in p_1 and y_1 satisfies (1), as well as all the translates gy_1 of y_1, for g in Z^2. Let us consider the S_{M_1}-P-name of y_1 : $H_s^{(1)}$. Let us now divide every tower U_s into t^2 $(t = |P|)$ subtowers $U_s^{(1,1)}, U_s^{(1,2)}, \ldots, U_s^{(i,j)}$ with positive measure and size s. Suppose furthermore, as before, that all the bases of the $U_s^{(i,j)}$ are in some uniform partition P_{i_1}. Let us fix (i, j) and look at $U_s^{(i,j)}$. Around the lower left corner of $U_s^{(i,j)}$ (that is, the levels in $U_s^{(i,j)}$ that are around the lower left level of $U_s^{(i,j)}$ on Fig. 2), we will change the P-name and replace it by $H_i^{(1)}$, that is, we change names in a square of size $2M_1 + 1$. The same way, around the upper

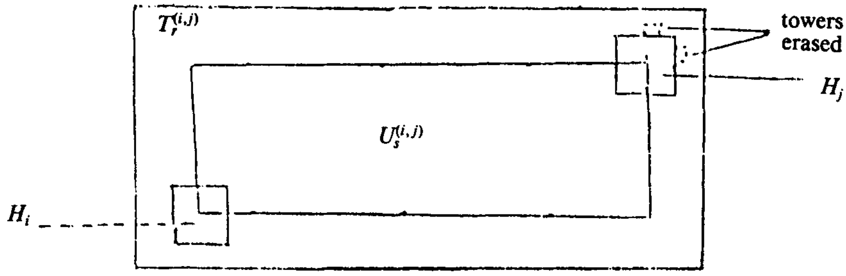

Fig. 2.

right corner, we change the P-name and replace it by $H_j^{(1)}$. The size of s ensures that this is possible.

We identify from now on $H_i^{(1)}$ and $H_j^{(1)}$ with the subset of T_r, where they have been put. This ensures:

$$\lambda(sp_i'^{(1)} \cap p_j'^{(1)}) > 0. \tag{2}$$

This gives the partition P_1'. Because $\lambda(\bigcup_r T_r) \leqq \alpha_1/2$ it is clear that $d(P, P_1') \leqq \alpha_1/2$. Fix

$$2\gamma_1 = \operatorname*{Inf}_{\substack{1 \leqq i,j \leqq t \\ s \text{ between } l_1 \text{ and } l_2}} \lambda(sp_i'^{(1)} \cap p_j'^{(1)}).$$

$\gamma_1 > 0$ by the construction and, in further steps, all the modifications made to P_1' will be much smaller that γ_1 so that, at the end, $d(\bar{P}, P_1') < \gamma_1$ and so $\bar{P}$ remains $(1, l_1, l_2)$ mixing. P_1' is still $(2, n_2, \alpha_2/5)$ uniform, if M_1^2/n_2 was chosen small enough: For any x, the S_{n_2}-P_1'-name of x and the S_{n_2}-P_0 name of x only differ in at most $8M_1^2$ places. Let us fix some x in Y. If the S_{M_1}-P_1'-name of x is diferent from its S_{M_1}-P-name this means, by our construction, that for some unique i, $S_{M_1}x \cap H_i'^{\,c} \neq \varnothing$ and $S_{M_1}x \cap H_i' \neq \varnothing$. Let us cover S_{M_1} by translates of S_{n_1}. Let us look at one of these translates: $S_{n_1}b$. Because P is $(1, n_1, \alpha_1/10)$ uniform, the distribution of the P_1'-name in $S_{n_1}bx$ is also up to $\alpha_1/10$ the true distribution of P_1', except in those translates such that $S_{n_1}bx \cap H_i^{(1)} \neq \varnothing$ and $S_{n_1}bx \cap H_i^{(1)c} \neq \varnothing$. (If $S_{n_1}bx \cap H_i^{(1)} = \varnothing$, the distribution of P_1' inside $S_{n_1}bx$ is the same as that of P in $S_{n_1}bx$ and if $S_{n_1}bx \subset H_i^{(1)}$, the distribution of P_1' in $S_{n_1}bx$ is the distribution of P in some $S_{n_1}gy_i$, for g in Z^2, where y_i was chosen to define $H_i^{(1)}$ (see above).) Thus, the only translates $S_{n_1}bx$ where there can be a problem are those intersecting the boundary of $H_i^{(1)}$. We conclude that in the S_{M_1}-P-name of x, there are at most $4n_1^2 * M_1$ places where the distribution of P_1' is not as good as it was for P because we chose $n_1^2/M_1 \leqq \alpha_1/10$. We conclude that P_1' remains $(1, M_1, \alpha_1)$ uniform. Furthermore, because of our construction P_1' is a uniform partition (all the bases of the Rohlin towers where we changed names were supposed to be in some uniform partition P_{i_1}).

Step k: By induction, we suppose that P'_{k-1} is (j, I_j, I_{j+1}) mixing for any $j \leqq k-1$. Let:

$$2\gamma_{j-1} = \operatorname*{Inf}_{\substack{s \text{ between } I_j \text{ and } I_{j+1} \\ p_l^{(j)}, p_m^{(j)} \text{ in } \vee_{c \text{ in } S_{j-1}} cP'_j}} \lambda(sp'^{(j)}_l \cap p'^{(j)}_m).$$

P'_j was constucted in step j. We suppose that all the changes, after step j, are much smaller than γ_{j-1} so that at the end of our induction procedure $d(\bar{P}, P'_j) \leqq \gamma_{j-1}/|S_{j-1}|$ and so $\bar{P}$ will be also $(j-1, I_{j-1}, I_j)$ mixing. At step k, we know γ_j for $1 \leqq j \leqq k-1$, so that the changes are made much smaller than $\inf_{0 \leqq j < k-1} \gamma_j$. We will suppose this to be true for all the changes in this step, and will not come back to this requirement. We also assume that these mixing properties were obtained at step j, by changing names in some set of disjoint Rohlin towers (see step 1) and putting as new names some $H_i^{(j)}$, in a square of size $2M_{j-1}+1$ (see induction assumptions below). We will also suppose: P'_{k-1} is a uniform partition, thus P'_{k-1} is $(k+1, n_{k+1}, \alpha_{k+1}/10)$ uniform for some properly chosen n_{k+1} bigger than M_k. We chose also M_{k+1} so that $n^2_{k+1}/M_{k+1} \leqq \alpha_{k+1}/10$. For every $l, l<k$, we chose n_{l+2} such that $M_{l+1} \ll n_{l+2}$ and $M_{l+2} \gg n_{l+2}$ and the partition P'_l constructed at step l was $(l+2, n_{l+2}, \alpha_{l+2}/10)$ uniform. For a fixed l and a fixed x, in $S_{M_l}x$ there is at most one pair (i,j) such that $S_{M_l}x$ is at least $2M_{l-1}$ close to the boundary of $H_i^{(j),k-1}$ ($H_i^{(j),k-1}$ is a part of $H_i^{(j)}$ (defined at step j) that we defined at step $k-1$). From this fact it is easy to conclude as in step 1 that P'_{l-1} is also (l, M_l, α_l) uniform. (The measure of $\vee_{c \text{ in } S_l} cP'_l$ is almost the same as that of $\vee_{c \text{ in } S_l} cP'_{k-1}$, and we will, from now on, suppose that this fact is included in other errors.) P'_{k-1} is also $(k, n_k, \alpha_k/5)$ uniform because P_{k-2} was $(k, n_k, \alpha_k/10)$ uniform and for any x, the S_{n_k}-P'_{k-2}-name of x differs in at most $2|S_{M_{k-1}}|$ places from the S_{n_k}-P'_{k-1}-name of x and n_k was chosen so that $2|S_{M_{k-1}}| \ll n_k$.

We choose $I_k = 10M_k$. Applying Lemma 2, with $\alpha = \alpha_k/2$, we find a sequence of disjoint Rohlin towers of size r: $T_r^{(n)}$ such that

$$\lambda\Bigg(\bigcup_{\substack{r \text{ between } I_{k-1}+10M_k \\ \text{and } I_k+10M_k}} T_r^{(n)}\Bigg) \leqq \alpha_k/2, \text{ for any } r \text{ between } I_{k-1}+10M_k \text{ and } I_k+10M_k.$$

Let us fix r between $I_{k-1}+10M_k$ and I_k+10M_k such that if $r=(r_1,r_2)$, $\mathrm{Min}(r_1,r_2) \geqq 10M_k$, we write $r=(s_1+10M_k, s_2+10M_k)$ where $s=(s_1,s_2)$ is between I_{k-1} and I_k. In $T_r^{(n)}$, we find easily a tower $U_s^{(n)}$ of size s whose boundary is at least $2M_k$ apart from that of $T_r^{(n)}$. The towers are chosen as in step 1, so that their bases are in some uniform partition P'_{i_k}. As in step 1, we further assume that any subtower we will choose in $U_s^{(n)}$ will also have its base in P'_{i_k}.

Our main work will be to improve the mixing properties of P'_{k-1} without losing too much of its uniformity properties.

Let $\bigvee_{c \text{ in } S_k} cP_k = (p_1^{(k)}, \ldots, p_{t_k}^{(k)})$, where for each l, $1 \leqq l \leqq t_k = |\bigvee_{c \text{ in } S_k} cP_k|$, $\lambda(p_l^{(k)}) > 0$. Let l, $1 \leqq l \leqq t_k$ be fixed. Because $\lambda(p_l^{(k)}) > 0$ and because almost every y in Y satisfies (1) for every j, $j \leqq k$, for $\bigvee_{c \text{ in } S_k} cP_k$, M_j and α_j (see induction hypothesis), there exists $y_l^{(k)}$ such that $y_l^{(k)}$ satisfies all these conditions and $y_l^{(k)}$ is in $p_l^{(k)}$. Let us consider the S_{M_k}-P_{k-1}-$\bigvee_{j=1}^{k} \bigvee_i H_i^{(j),k}$-name of $y_l^{(k)}$: $H_l'^{(k)}$, it is an abstract name; the following work is done on it even if we explain it as if we were working on $S_{M_k} y_l^{(k)}$ (see step 1). If there is a "tower" $H_i^{(k-1),k-1}$ whose boundary is less than $2M_{k-2}$ apart from the boundary of $H_l'^{(k)}$ and has some part of it in $H_l'^{(k)}$, we give to those places in $H_l'^{(k)}$ their previous P'_{k-2}-name. Doing so, we may give birth to towers or part of towers $H_i^{(k)}$ that were erased at step $k-1$. Now we look to towers $H_i^{(k-2),k-1}$ (and the part that came back) and consider the part in those towers whose boundary is less than $2M_{k-3}$ apart from those of $H_l^{(k)}$; we give to the corresponding places their previous P'_{k-3}-name. We go on with this process until we arrive at towers $H_i^{(1),k-1}$ (and the part that came back), we change names in $H_l^{(k)}$ for those places that intersect the boundary of $H_i^{(1),(k-1)}$: We thus obtain $H_l^{(k)}$. Let us now divide the tower $U_s^{(k)}$ into t_k^2 ($t_k = |\bigvee_{c \text{ in } S_k} cP_k|$) subtowers: $U_s^{(1),(1)}, U_s^{(1),(2)}, \ldots, U_s^{(t_k),(t_k)}$, with positive measure and size s. Let us fix (i, j) and look at $U_s^{(i),(j)}$:

Around the lower left corner of $U_s^{(i),(j)}$ (in a square of size $2M_{k-1}+1$), we will replace the P_k-name by $H_i^{(k)}$. The same way around the upper right corner, we will replace the P_k-name by $H_j^{(k)}$. The size of s (s is between $I_{k-1} + 10M_k$ and I_k) enables us to do so. This way we ensure $\lambda(sp_i'^{(k)} \cap p_j'^{(k)}) > 0$, for any s between I_{k-1} and I_k and any two atoms of $\bigvee_{c \text{ in } S_k} cP_k$. We have thus defined all our towers $H_i^{(k)}$, for $j \leqq k$.

If in $U_s^{(i),(j)}$, or more precisely in the part $T_r^{(i),(j)}$ in $T_r^{(k)}$ corresponding to the images of $U_s^{(i),(j)}$, there are towers $H_i^{(k-1),k-1}$, whose boundary is less than $2M_{k-2}$ apart from that of $H_i^{(k)}$ (or $H_j^{(k)}$), we delete them from $H_i^{(k-1),k-1}$, obtain $H_i^{(k-1),k}$ and give to the deleted part its previous P_{k-1}-name. We go on then with the process of "putting on and off" towers around $H_i^{(k)}$ (or $H_j^{(k)}$). We finally obtain the partition P'_k. Because $\lambda(\bigcup_r T_r^{(k)}) \leqq \alpha_k/2$, it is clear that $d(P_k, P'_k) \leqq \alpha_k/2$. It also becomes clear that P'_k is now $(k-1, I_{k-1}, I_k)$ mixing.

We let $2\gamma_{k-1} = \mathrm{Inf}_{0<i,j<t_k} \lambda(sp_i'^{(k)} \cap p_j'^{(k)})$ and $\gamma_{k-1} > 0$. It is easy to see that a S_M-P_k'-name differs from a S_M-P_k-name only when it is in a given tower $T_r^{(k)}$ and in this case it differs only in at most $10M_{k-1}$ places so that P_k' remains $(k+1, M_{k+1}, \alpha_k/5)$ uniform

Let us check that we can go on with our induction: Let us consider the S_{M_j}-P_k-name of some x in Y, for $j<k$. This name is identical to the S_{M_j}-P_j-name of some y in Y except if $S_{M_j}x$ is closer than $2M_{j-1}$ from the boundary of some $H_i^{(l),k}$ for $l>k$. For l fixed, it is clear that this can happen for only one i (from the construction). Suppose now that it would be close to the boundary of two towers $H_i^{(l),k}$ and $H_{i'}^{(l'),k}$ for $j<l<l'$. Now the boundary of $H_i^{(l),k}$ would be closer than $M_j + 4M_{j-1} \leqq 2M_{l-1}$ from that of $H_{i'}^{(l'),k}$ and this is prevented by our construction. This property enables us to keep our uniform properties and go on with our induction. Suppose now that we chose n_i and M_i so that $\Sigma_{i=2}^{i=+\infty} 10M_{i-1}/n_i < +\infty$. For almost every y in Y, for any i, because of the Borel–Cantelli lemma, there is an index j so that we do not change numerations of the atoms in the S_{M_i}-name of y after step j. This proves that $\bar{P}$ is also (i, M_i, α_i) uniform and finishes our proof of Theorem 3.

REMARK. In the case of a general amenable group, the proof is almost similar except that, instead of using as basic shapes squares in Z^2, we will use tiling sets, the same way as in[R1].

ACKNOWLEDGEMENT

Professor B. Weiss asked us the question we answered in this paper. This work was done under his guidance and constant help.

REFERENCES

[H–R] G. Hansel and J. P. Raoult, *Ergodicity, uniformity and unique ergodicity*, Indiana Univ. Math. J. **23** (1974), 221–237.

[J] R. I. Jewett, *The prevalence of uniquely ergodic systems*, J. Math. Mech. **19** (1970), 717–729.

[K] W. Krieger, *On unique ergodicity*, Proc. Sixth Berkeley Symposium on Math. Stat. and Prob., 1970, pp. 327–346.

[L] E. Lehrer, *Topological mixing and uniquely ergodic systems*, Isr. J. Math. **57** (1987), 239–255.

[R1] A. Rosenthal, *Strictly ergodic models and amenable group actions*, preprint.

[R2] A. Rosenthal, *Strictly ergodic models for commuting ergodic transformations*, preprint.

[W] B. Weiss, *Strictly ergodic models for dynamical systems*, Bull. Am. Math. Soc. **13** (1985), 143–146.